EVADING THE INFRARED PROBLEM OF THERMAL QCD*

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Due to asymptotic freedom, QCD is guaranteed to be accessible to perturbative methods at asymptotically high temperatures. However, in 1979 Linde has pointed out the existence of an "infrared wall", beyond which an infinite number of Feynman diagrams contribute. Following a proposal by Braaten and Nieto, it is shown explicitly how the limits to computability that this infrared problem poses can be overcome in the framework of dimensionally reduced effective theories.

1. Introduction

The theory of strong interactions, Quantum Chromodynamics (QCD), is guaranteed to be accessible to perturbative methods once one of its parameters, the temperature T, is increased to asymptotically high values. This statement relies solely on the well-known property of asymptotic freedom[1].

In practice, however, calculations of corrections to the behavior of an ideal gas of quarks and gluons, the limit that is formally realized at infinite T, are obstructed by infrared divergences[2]: For every observable one sets out to compute, there exists an order of the perturbative expansion to which an infinite number of Feynman diagrams contribute ("infrared wall").

No method is known how to re-sum these infinite classes of diagrams, a fact that seriously obstructs progress in the field of thermal QCD, a field that presently receives attention particularly due to its relevance to the ongoing program of heavy-ion collisions at RHIC, where one of the main focuses is to explore the phase diagram of QCD.

For the QCD pressure – as an example of a static thermodynamic observable – it has been shown explicitly how the limits to computability that the infrared problem poses can be overcome in the framework of dimensionally reduced effective theories[3,4,5]. The key idea is to measure the effect of ultra-soft ("magnetic") gluons by lattice Monte–Carlo simulations in 3–dimensional (3d) pure gauge theory (MQCD), and to match this theory up

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to full thermal QCD in perturbation theory, via 3d gauge + adjoint Higgs theory (EQCD).

The expansion of the QCD pressure in the effective theory framework, up to the order where infrared contributions are relevant, is now known analytically[5]. The $\mathcal{O}(g^6)$ coefficient is non–perturbative, but *computable*. All other effects (N_f- or μ_f-dependence; orders g^7, ...) are perturbative.

2. Status of the QCD Pressure

Below, we specify the contributions to the pressure $p_{\rm QCD} = p_G + p_M + p_E$ from each physical scale individually, for the slightly more general case of gauge group SU(N_c) and N_f quark flavors, mainly following Ref. [5]. We will work at zero quark masses $m_{q_i} = 0$ and vanishing chemical potential $\mu_f = 0$, and display all dependence on the $\overline{\rm MS}$ scale by $L \equiv \ln \frac{\bar\mu}{4\pi T}$.

- Contributions from the ultra-soft scale g^2T, i.e. from MQCD:

$$\frac{p_G(T)}{\mu^{-2\epsilon}} = d_A 16\pi^2 T^4 \hat{g}_M^6 \left[\alpha_G\left(\frac{1}{\epsilon} + 8L - 8\ln(8\pi\hat{g}_M^2)\right) + \beta_G + \mathcal{O}(\epsilon)\right], \quad (1)$$

where $d_A = N_c^2 - 1$, $\alpha_G = \frac{43}{96} - \frac{157}{6144}\pi^2$ is a perturbative 4-loop coefficient, and β_G is non-perturbative, requiring two ingredients: a lattice-measurement in MQCD, and a perturbative computation which allows to match between the lattice and the continuum regularization schemes. For the latter, a 4-loop lattice-regularized computation is needed, which could possibly be accomplished with methods used in Refs. [6,7]. The matching condition reads

$$\hat{g}_M^2 \equiv \frac{N_c g_M^2}{16\pi^2 T} = \hat{g}_E^2 + \mathcal{O}(\hat{g}_E^4 \hat{m}_E^{-1}) \; . \quad (2)$$

- Contributions from the soft scale gT, i.e. from EQCD:

$$\begin{aligned}\frac{p_M(T)}{\mu^{-2\epsilon}} = d_A 16\pi^2 T^4 \Bigg\{ & \hat{m}_E^3\left[\frac{1}{3} + \mathcal{O}(\epsilon)\right] \\ & + \hat{g}_E^2 \hat{m}_E^2 \left[-\frac{1}{4\epsilon} + \left(-L + \frac{1}{2}\ln\hat{m}_E^2 + \ln 2 - \frac{3}{4}\right) + \mathcal{O}(\epsilon)\right] \\ & + \hat{g}_E^4 \hat{m}_E \left[\left(-\frac{89}{24} - \frac{\pi^2}{6} + \frac{11}{6}\ln 2\right) + \mathcal{O}(\epsilon)\right] \\ & + \hat{g}_E^6\left[\alpha_M\left(\frac{1}{\epsilon} + 8L - 4\ln\hat{m}_E^2 - 8\ln 2\right) + \beta_M + \mathcal{O}(\epsilon)\right] \\ & + \hat\lambda_E^{(1)}\hat{m}_E^2\left[\frac{\hat{n}-2}{4} + \mathcal{O}(\epsilon)\right] + \hat\lambda_E^{(2)}\hat{m}_E^2\left[\frac{1-3\hat{n}}{4} + \mathcal{O}(\epsilon)\right] \\ & + \mathcal{O}(\hat{g}_E^8\hat{m}_E^{-1}, \hat\lambda_E^2\hat{m}_E)\Bigg\} \, , \end{aligned} \quad (3)$$

with $\hat n \equiv \frac{N_c^2-1}{N_c^2}$, the 4-loop coefficients $\alpha_M = \frac{43}{32} - \frac{491}{6144}\pi^2$, $\beta_M \approx -1.391512$ and the matching parameters

$$\hat m_E^2 \equiv \left(\frac{m_E}{4\pi T}\right)^2 = \hat g^2 \left[\tilde\alpha_{\rm E4} + (2\tilde\alpha_{\rm E4}L + \tilde\alpha_{\rm E5})\,\epsilon + \mathcal{O}(\epsilon^2)\right] \\ +\hat g^4 \left[\left(2\hat\beta_0\tilde\alpha_{\rm E4}L + \tilde\alpha_{\rm E6}\right) + \bar\beta_{\rm E2}\epsilon + \mathcal{O}(\epsilon^2)\right] + \mathcal{O}(\hat g^6)\,, \quad (4)$$

$$\hat g_E^2 \equiv \frac{N_c g_E^2}{16\pi^2 T} = \hat g^2 + \hat g^4 \left[\left(2\hat\beta_0 L + \tilde\alpha_{\rm E7}\right) + \bar\beta_{\rm E3}\epsilon + \mathcal{O}(\epsilon^2)\right] + \mathcal{O}(\hat g^6)\,, \quad (5)$$

$$\hat\lambda_E^{(1)} \equiv \frac{N_c^2\lambda_E^{(1)}}{16\pi^2 T} = \hat g^4 \left[4 + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\hat g^6)\ , \quad (6)$$

$$\hat\lambda_E^{(2)} \equiv \frac{N_c\lambda_E^{(2)}}{16\pi^2 T} = \hat g^4 \left[\frac{4}{3}(1-z) + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\hat g^6)\ , \quad (7)$$

where, for brevity, $z \equiv N_f/N_c$ and

$$\tilde\alpha_{\rm E4} = \frac{2+z}{6}\ , \quad \tilde\alpha_{\rm E6} = \frac{1}{3}\tilde\alpha_{\rm E4}(6\hat\beta_0\gamma + 5 + 2z - 8z\ln 2) - \frac{z}{2}\hat n\ , \quad (8)$$

$$\tilde\alpha_{\rm E5} = 2\tilde\alpha_{\rm E4}Z_1 + \frac{z}{6}(1-2\ln 2)\ , \quad \tilde\alpha_{\rm E7} = 2\hat\beta_0\gamma + \frac{1}{3} - \frac{8}{3}z\ln 2\ , \quad (9)$$

and $\bar\beta_{\rm E2}$ (see Sec. 3) and $\bar\beta_{\rm E3}$ remain to be computed by perturbatively matching suitable correlators computed in thermal QCD and in EQCD.

- Contributions from the hard scale $2\pi T$, i.e. from thermal QCD:

$$\frac{p_E(T)}{\mu^{-2\epsilon}} = d_A 16\pi^2 T^4 \frac{1}{16}\frac{1}{45}\left\{\left[1 + \frac{7}{4}\frac{z}{\hat n}\right] + \hat g^2\left[\tilde\alpha_{\rm E2} + \mathcal{O}(\epsilon)\right]\right. \\ +\hat g^4\left[\tilde\alpha_{\rm E4}\frac{180}{\epsilon} + (180\cdot 6\tilde\alpha_{\rm E4} + 2\hat\beta_0\tilde\alpha_{\rm E2})L + \tilde\alpha_{\rm E3} + \mathcal{O}(\epsilon)\right] \\ \left. +\hat g^6\left[\frac{\tilde\beta_{\rm E1}^{(\rm div)}}{\epsilon} + \tilde\beta_{\rm E1}^{(L^2)}L^2 + \tilde\beta_{\rm E1}^{(L)}L + \tilde\beta_{\rm E1} + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\hat g^8)\right\}, \quad (10)$$

with $\tilde\alpha_{\rm E2} = -\frac{5}{4}(4+5z)$ and, writing $Z_1 \equiv \frac{\zeta'(-1)}{\zeta(-1)}$ and $Z_3 \equiv \frac{\zeta'(-3)}{\zeta(-3)}$,

$$\tilde\alpha_{\rm E3} = 180(\tilde\alpha_{\rm E4})^2\gamma + 5\left[\left(\frac{116}{5} + \frac{220}{3}Z_1 - \frac{38}{3}Z_3\right)\right. \\ +\frac{z}{2}\left(\frac{1121}{60} - \frac{157}{5}\ln 2 + \frac{146}{3}Z_1 - \frac{1}{3}Z_3\right) \\ \left. +\frac{z^2}{4}\left(\frac{1}{3} - \frac{88}{5}\ln 2 + \frac{16}{3}Z_1 - \frac{8}{3}Z_3\right) + \frac{z}{4}\hat n\left(\frac{105}{4} - 24\ln 2\right)\right], \quad (11)$$

and unknown coefficients $\beta_{\rm E1}$, which can be determined e.g. by a 4-loop computation of vacuum diagrams in thermal QCD. Since $p_{\rm QCD}$ is physical,

the divergent and scale-dependent parts of $\beta_{\rm E1}$ are related to the other coefficients introduced in the above, serving as a valuable check on this open computation. Specifically, from 2-loop running of the 4d gauge coupling

$$\hat{g}^2 \equiv \frac{N_c g^2(\bar\mu)}{16\pi^2} = \hat{g}^2(\bar\mu_0) + \hat{g}^4(\bar\mu_0)(-2\hat\beta_0\ell) + \hat{g}^6(\bar\mu_0)(4\hat\beta_0^2\ell^2 - 2\hat\beta_1\ell) \quad (12)$$

with beta-function coefficients $\hat\beta_0 = \frac{11-2z}{3}$, $\hat\beta_1 = \frac{34}{3} - \frac{10}{3}z - z\hat{n}$, $\overline{\rm MS}$ scale parameter $\bar\mu^2 = 4\pi e^{-\gamma}\mu^2$ and $\ell \equiv \ln\frac{\bar\mu}{\bar\mu_0} = L - \ln\frac{\bar\mu_0}{4\pi T}$, one can fix

$$\tilde\beta_{\rm E1}^{(\rm div)} = 180\left[4\hat\beta_0\tilde\alpha_{\rm E4}L + \tilde\alpha_{\rm E6} + \tilde\alpha_{\rm E4}\tilde\alpha_{\rm E7} - 4(\alpha_G + \alpha_M)\right], \quad (13)$$

$$\tilde\beta_{\rm E1}^{(L^2)} = 180\left[20\hat\beta_0\tilde\alpha_{\rm E4} + \tilde\beta_{\rm E2}^{(L^2)} + \tilde\alpha_{\rm E4}\tilde\beta_{\rm E3}^{(L^2)}\right] + 4\hat\beta_0^2\tilde\alpha_{\rm E2}\,, \quad (14)$$

$$\tilde\beta_{\rm E1}^{(L)} = 180\left[4\tilde\alpha_{\rm E6} + 6\tilde\alpha_{\rm E4}\tilde\alpha_{\rm E7} - 2\hat\beta_0\tilde\alpha_{\rm E5} - 32(\alpha_G + \alpha_M) + \tilde\beta_{\rm E2}^{(L)} + \tilde\alpha_{\rm E4}\tilde\beta_{\rm E3}^{(L)}\right] + 2\hat\beta_1\tilde\alpha_{\rm E2} + 4\hat\beta_0\tilde\alpha_{\rm E3}\,. \quad (15)$$

3. Determination of β_{E2}

To determine $\bar\beta_{E2}$ in Eq. (4), one can e.g. match the pole masses of the A_0 propagator. In thermal QCD, writing $\Pi_{00}^{ab}(k_0=0,\vec{k}) = \delta^{ab}\Pi(k^2)$, one needs to solve $k^2 + \Pi(k^2) = 0$ at $k^2 = -m_{\rm pole}^2$. Inserting a loop expansion for the self-energy Π and noting that the leading-order solution gives a perturbatively small $m_{\rm pole}^2 \sim g^2$, one can Taylor-expand to get

$$\hat{m}_{\rm pole}^2 \equiv \left(\frac{m_{\rm pole}}{4\pi T}\right)^2 = \hat\Pi_1(0) + \hat\Pi_2(0) - \hat\Pi_1(0)\Pi_1'(0) + \mathcal{O}(\hat{g}^6)\,. \quad (16)$$

The bare gluon self-energies can be deduced from the literature[3] as

$$\hat\Pi_1(0) \equiv \frac{\Pi_1(0)}{16\pi^2T^2} = \hat{g}_B^2\left[(d-2)^2\hat{I}_b(1) - 2z(d-2)\hat{I}_f(1)\right], \quad (17)$$

$$\Pi_1'(0) = \hat{g}_B^2\left[\frac{1}{6}(-22+7d-d^2)\hat{I}_b(2) + \frac{z}{3}(d-2)\hat{I}_f(2)\right], \quad (18)$$

$$\hat\Pi_2(0) \equiv \frac{\Pi_2(0)}{16\pi^2T^2} = \hat{g}_B^4\left[\left(2z\hat{I}_f(1) - (d-2)\hat{I}_b(1)\right)(12-8d+d^2)\hat{I}_b(2) + \hat{n}z(d-4)(d-2)\left(\hat{I}_b(1) - \hat{I}_f(1)\right)\hat{I}_f(2)\right], \quad (19)$$

where we have used scaled bosonic and fermionic 1-loop tadpole integrals $\hat{I}(x) = 16\pi^2(4\pi T)^{2x-4}T\sum_{n=-\infty}^{\infty}\mu^{2\epsilon}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{(\omega_n^2+\vec{p}^2)^x}$, with $\omega_n = 2n\pi T$ for bosons and $(2n+1)\pi T$ for fermions,

$$\hat{I}_b(x) = \mu^{2\epsilon}\frac{2^{2x-3}}{\sqrt\pi}(\sqrt\pi T)^{d-4}\zeta(1+2x-d)\frac{\Gamma(x+\frac{1-d}{2})}{\Gamma(x)} \quad (20)$$

and correspondingly $\hat{I}_f(x) = (2^{2x+1-d} - 1)\hat{I}_b(x)$.

In EQCD, the solution of $k^2 + m_E^2 + \Pi_E(k^2) = 0$ at $k^2 = -m_{\rm pole}^2$ is simply $m_{\rm pole}^2 = m_E^2$, since again treating $m_{\rm pole}^2$ as perturbatively small and Taylor-expanding, there is no scale left in Π_E, such that it vanishes in $\overline{\rm MS}$. Renormalizing Eq. (16) via $\hat{g}_B^2 = Z_{\hat{g}}^2 \hat{g}^2$ with $Z_{\hat{g}}^2 = 1 - \hat{g}^2\hat{\beta}_0/\epsilon + \mathcal{O}(\hat{g}^4)$ and comparing with Eq. (4), one reproduces Eqs. (8–9) and finally obtains the coefficients in $\bar{\beta}_{\rm E2} = 6\hat{\beta}_0\tilde{\alpha}_{\rm E4}L^2 + \tilde{\beta}_{\rm E2}^{(L)}L + \tilde{\beta}_{\rm E2}$, which are

$$\begin{aligned}\tilde{\beta}_{\rm E2}^{(L)} &= 4\hat{\beta}_0\tilde{\alpha}_{\rm E4}(2\gamma + Z_1) + \frac{1}{9}(20 + 29z + 2z^2) \\ &\quad - 2z\,(\hat{n} + 3\ln 2) - \frac{4}{3}z^2 \ln 2\;, \qquad (21)\\ \tilde{\beta}_{\rm E2} &= \frac{1}{4}\hat{\beta}_0\tilde{\alpha}_{\rm E4}\left(16\zeta'(1) + \pi^2\right) + \frac{2}{3}\tilde{\alpha}_{\rm E4}Z_1(6\hat{\beta}_0\gamma + 5 + 2z - 8z\ln 2) \\ &\quad + \frac{2}{9}\gamma(5 + 10z - (19 + 2z)z\ln 2) + \frac{2}{9} + \frac{z}{18}(7 + 6\ln 2 - 16\ln^2 2) \\ &\quad + \frac{z^2}{9}(1 - 2\ln 2 + 4\ln^2 2) - \frac{z}{6}\hat{n}(3 + 6\gamma + 6Z_1 + 10\ln 2)\;. \qquad (22)\end{aligned}$$

4. Outlook

The future should see a completion of the above setup, thereby establishing a first example of successfully computing an observable beyond the infrared wall. Once this is achieved, thermal QCD in its high-temperature phase will again be amenable to perturbative calculations, opening up numerous opportunities to precisely compute observables that might become relevant to the RHIC program, to future accelerators, and to cosmology. Indeed, the next term in the series, formally of order $\mathcal{O}(g^7)$, requires the corrections of order $\mathcal{O}(\hat{g}_E^4/\hat{m}_E)$ to Eq. (2) (known), of order $\mathcal{O}(\hat{g}_E^8/\hat{m}_E)$ to Eq. (3) (5-loop vacuum diagrams in EQCD), and of order $\mathcal{O}(\hat{g}^6)$ to Eq. (4) (3-loop 2-point functions in thermal QCD), and will then certainly be within reach.

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